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**FACULTY WORKING
PAPER NO. 782**

**The Effects of Transaction Costs and Different
Borrowing and Lending Rates on the
Option Pricing Model**

*John E. Gilster
William Lee*

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The Effects of Transaction Costs and
Different Borrowing and Lending Rates on the
Option Pricing Model

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Abstract

The original Black-Scholes option pricing model assumes zero transaction costs and borrowing and lending at the risk free rate. This paper relaxes both assumptions and demonstrates that most options have a range of legitimate equilibrium values and that this range frequently fails to include the traditional Black-Scholes value.

A more startling discovery is the observation that under some conditions there may be no equilibrium option price. Instead there may be a bounded disequilibrium within which a single option will offer a risk free return above the Treasury bill rate while simultaneously permitting borrowing below the market borrowing rate.

I. Introduction

The Black-Scholes model requires an investor to create a risk free hedge by taking a position in a common stock and the opposite position in the stock's underlying option. The stocks and options are held in proportions such that any price movement in the stock is perfectly offset by an opposite movement in the option. These proportions are readjusted continuously throughout the life of the hedge. The hedge is therefore risk free and yields the risk free rate.

If the hedge consists of a long position in common stock and a short position in options the hedge will require a net investment on which the investor will earn the risk free rate (an "investment hedge"). If the hedge consists of a long position in an option and a short position in the stock, the hedge supplies funds to the investor for which he pays interest (a "borrowing hedge").¹

The original Black-Scholes (1972) option pricing model assumes zero transaction costs and implicitly assumes borrowing and lending at the risk free rate. Under these assumptions the option price appropriate to an investment hedge is equal to the option price appropriate to a borrowing hedge and this determines a unique equilibrium option price. This will be shown to be a special case of a more general model.

If transaction costs are ignored, the effects of different borrowing and lending rates are relatively obvious. The option price appropriate to an investment hedge is the traditional Black and Scholes price (and therefore offers the risk free rate) and the option price appropriate to a borrowing hedge can be calculated from the Black-Scholes model equation with the market borrowing rate substituted for the Treasury bill rate (the hedge therefore costs the borrowing rate). Obviously, the option price appropriate to a borrowing hedge will be greater than the option price appropriate to an investment hedge.

When transaction costs are considered, option prices must be adjusted so as to earn the investor the risk free rate on an investment hedge or cost him the borrowing rate on a borrowing hedge net of transaction costs.

In essence, the profit from an investment hedge comes from the declining time premium of an option that has been short sold (or "written"): The additional revenue needed to pay transaction costs can be generated by raising the initial option price to provide for a greater decline in time premium and a greater profit for the hedge's short position in options.

In essence, the cost of a borrowing hedge results from the deteriorating time premium of the hedge's long position in options. To provide for borrowing at the market rate after transaction costs the initial option price must be reduced so as to provide for less deterioration in time premium and more funds available to pay transaction costs.

The reader will note that if transaction costs and the borrowing and lending rate spread are of precisely the right size, the transaction cost adjusted option price for an investment hedge can equal the transaction cost adjusted option price for a borrowing hedge. In this case the market imperfection of different borrowing and lending rates and the market imperfection of positive transaction costs cancel to produce a unique equilibrium option price.

This precise cancellation is, of course, rare. Usually one of the two imperfections dominates yielding a bounded range of option prices. Section IV and the conclusion of the paper point out the potentially bizarre nature of some of these situations.

Section I of the paper presents an introduction and a general description of the problem. Section II derives a continuous time

option pricing model which includes rebalancing transaction costs. Section III suggests ways in which some investors may reduce the cost of acquiring and terminating the hedge position. Section IV calculates equilibrium option prices with transaction costs and different borrowing and lending rates and describes the sometimes bizzare nature of its findings. Section V presents a conclusion and summary.

II. Transaction Costs of Hedge Rebalancing

The Black-Scholes model assumes that the price of an option, $w(x,t)$, is a function of stock price x , and time, t . In this case, the equity in an investing hedge of one stock share long and $n = 1/w_1$ options short is $x - wn$ (where the subscript refers to the partial derivative of $w(x,t)$ with respect to its first argument). The equity change in a short interval Δt can be expressed as:

$$\Delta(x - w/w_1) = \Delta x - \Delta(wn) \quad (1)$$

$$= \Delta x - [w(x+\Delta x, t+\Delta t)n(x+\Delta x, t+\Delta t) - w(x,t)n(x,t)]$$

which can be expanded to:

$$= \Delta x - \{[w + w_1 \Delta x + \frac{1}{2} w_{11} (\Delta x)^2 + w_2 \Delta t][n + n_1 \Delta x + \frac{1}{2} n_{11} (\Delta x)^2 + w_2 \Delta t] - wn\}$$

Substituting $v^2 = (\Delta x/x)^2/\Delta t$ and $\delta = (\Delta x/x)/\Delta t$, and keeping only the terms of Δx and Δt , equation (1) becomes:

$$\Delta(x - w/w_1) = (-\frac{1}{2} v^2 x^2 w_{11} - w_2) n \Delta t + \quad (2)$$

$$(-\delta x w n_1 - w n_2 - \frac{1}{2} v^2 x^2 w n_{11} - v^2 x^2 w_1 n_1) \Delta t$$

The first term on the right side of equation (2) is the part of the equity change which yields the risk-free rate as derived in the Black and Schole model:

$$\begin{aligned}
 \Delta x - \Delta w/w_1 &= \Delta x - [w(x+\Delta x, t+\Delta t) - w(x, t)]/w_1 \\
 &= \Delta x - [w + w_1 \Delta x + \frac{1}{2} w_{11} (\Delta x)^2 + w_2 \Delta t - w]/w_1 \\
 &= (\frac{1}{2} w_{11} x^2 v^2 \Delta t - w_2 \Delta t)/w_1 \\
 &= (\frac{1}{2} v^2 x^2 w_{11} - w_2) \Delta t
 \end{aligned} \tag{3}$$

The second term of equation (2) is the extra capital required to maintain the hedge position.² The extra capital is composed of the change in the number of options Δn at the changed price $-w(x+\Delta x, t+\Delta t)$ i.e., $-w(x+\Delta x, t+\Delta t) \Delta n$

$$\begin{aligned}
 &= -[w + w_1 \Delta x + \frac{1}{2} w_{11} (\Delta x)^2 + w_2 \Delta t] [n + n_1 \Delta x + \frac{1}{2} n_{11} (\Delta x)^2 + n_2 \Delta t - n] \\
 &= (-\delta x w n_1 - w n_2 - \frac{1}{2} v^2 x^2 w n_{11} - v^2 x^2 w_1 n_1) \Delta t
 \end{aligned} \tag{4}$$

Accompanying the extra capital, the transaction cost by which the equity change should be reduced is $\alpha | -w(x+\Delta x, t+\Delta t) \Delta n |$, where α is the transaction cost rate for options. Therefore, the equity change $\Delta x - \Delta w/w_1$ yields the risk-free investing rate r^1 on the equity $x - w/w_1$ after the cost, $\alpha | -w(x+\Delta x, t+\Delta t) \Delta n |$, has been deducted. Therefore:

$$\Delta x - \Delta w/w_1 - \alpha | -w(x+\Delta x, t+\Delta t) \Delta n | = (x - w/w_1) r^1 \Delta t \tag{5}$$

Substituting equations (3) and (4) into equation (5) and replacing n , n_1 , n_{11} and n_2 by $1/w_1$, $-w_{11}/w_1^2$, $(2w_{11}^2 - w_1 w_{111})/w_1^3$ and $-w_{12}/w_1^2$ respectively, yields:

$$r^i w - r^i_{xw_1} - \frac{1}{2} v^2 x^2 w_{11} - w_2 = \alpha g \quad (6)$$

where

$$\begin{aligned} g &= w_1 \left| -\delta x w n_1 - w n_2 - \frac{1}{2} v^2 x^2 w_{11} - v^2 x^2 w_1 n_1 \right| \\ &= \left| \delta x w w_{11} / w_1 + w w_{12} / w_1 + v^2 x^2 (-w w_{11}^2 / w_1^2 + \frac{1}{2} w w_{111} / w_1 + w_{11}) \right| \quad (7) \end{aligned}$$

Equation (6) is the differential equation for the option price from an investing hedge. Similarly, one could be derived from a borrowing hedge, i.e.,

$$r^b w - r^b_{xw_1} - \frac{1}{2} v^2 x^2 w_{11} - w_2 = -\alpha g, \quad (8)$$

where r^b is the appropriate borrowing rate.

To solve for equation (6), it is reasonable to assume the solution w^i differs only slightly from the Black and Scholes model solution because the transaction cost rate, α , is quite small. Let w^0 be the Black and Scholes solution and $\alpha w'$ the correction, then

$$w^i = w^0 + \alpha w' \quad (w^0 \gg \alpha w') \quad (9)$$

Replacing w by w^i in equation (6):

$$r^i w' - r^i_{xw_1} - \frac{1}{2} v^2 x^2 w'_{11} - w'_2 = g \quad (10)$$

Since $w^i = w^0$, g can be approximated by

$$g \approx \left| \delta x w^0 w_{11}^0 / w_1^0 + w^0 w_{12}^0 / w_1^0 + v^2 x^2 (-w^0 w_{11}^0{}^2 / w_1^0{}^2 + \frac{1}{2} w^0 w_{111}^0 / w_1^0 + w_{11}^0) \right| \quad (11)$$

Because the hedge will not change when $t = t^*$, $x \rightarrow \infty$ or $x = 0$, there will be no transactions costs. Therefore, $w^i = w^0$ and

$$w'(x, t^*) = w'(\infty, t) = w'(0, t) = 0 \quad (12)$$

The same substitution for w' used in the Black and Scholes model yields:

$$w'(x, t) = e^{r^i(t^*-t)} Z(u, s)$$

where:

$$u = \frac{2}{v^2} \left(r^i - \frac{v^2}{2} \right) \left[\ln \frac{x}{c} - \left(r^i - \frac{v^2}{2} \right) (t-t^*) \right], \quad (14)$$

$$s = - \frac{2}{v^2} \left(r^i - \frac{v^2}{2} \right)^2 (t-t^*) \quad (15)$$

and c is the exercise price of the option.

Equation (15) implies:

$$t = t^* - sv^2 / (2(r^i - \frac{v^2}{2})^2)$$

Equation (14) implies:

$$x = c \exp \{ (u-s) / [r^i - \frac{v^2}{2}] \}$$

Substituting x and w' into equation (10) and (11):

$$Z_2 - Z_{11} = h \quad (16)$$

with $h = h(u, s)$ being the function g after multiplying by $e^{r^i(t^*-t)v^2/(2(r^i - \frac{v^2}{2})^2)}$ and substituting in x and t . The boundary conditions, equation (12), become:

$$Z(u,0) = Z(\infty,0) = Z(-\infty,0) = 0$$

The solution for equation (16) is given by Butkov (1968, pp. 525-526):

$$Z(u,s) = \int_0^s \int_{-\infty}^{\infty} \frac{e^{-(u-u')^2/4(s-s')}}{(4\pi(s-s'))^{1/2}} h(u',s') du' ds' \quad (17)$$

Substituting (17) into (13) and then (9), yields the solution w^i . Similarly, w^b can be solved for by using interest rate r^b and replacing equations (9) and (10) with:

$$w^b = w^0 + \alpha w' \text{ and} \quad (18)$$

$$r^b w' - r^b \times w'_1 - \frac{1}{2} v^2 \times w'_{11} - w'_2 = -g \quad (19)$$

III. Initiation and Termination Costs

In option hedging, the lowest cost market participant will usually be the investor for whom acquiring (or short selling) the common stock portion of the hedge is a by-product of other activities.

An investment hedge consists of a long position in common stock and a short position in options. An investor who owns but wishes to sell the common stock for which the hedge is to be written can form the hedge without incurring a marginal cost for buying or selling the stock. In this case, instead of selling the stock immediately the stock is retained and the usual hedged position of w/w_1 worth of options are written for each share of stock held. This hedge is held until either:

- 1) The option price drops below 1/16 point at which time C.B.O.E. trading in the option is halted. A hedge can no longer be formed and the common stock is (finally) sold.
- 2) The option is in the money at expiration and the stock is called away. In this case transaction costs are calculated as if the stock were sold at the exercise price, C rather than the actual stock price X^* .

As soon as the hedge is formed, stock price movements are neutralized and the stock used in the hedge is in effect sold. The initial cash flows (including the savings from not actually selling the stock) are:

$$\alpha w/w_1 - \alpha_x x$$

where α_x is the transaction cost rate for common stock transactions. When the stock is finally (actually) sold and the hedge position closed out the flows are:

$$\alpha_x \text{Min}(x^y; c) = \alpha_x (x^y - w^y) \quad (20)$$

where the superscript y indicates values at the time the hedge is closed out (not necessarily at expiration; see contingency 1 above).³

In equilibrium (ignoring dividends) the discounted present value of the expected value of x^y is x . Therefore, when the hedge is constructed the risk adjusted present value of the total cash outflows are:

$$\alpha w/w_1 - \alpha_x e^{-K\Delta t} E(w^y) \quad (21)$$

where K is the discount rate appropriate to the option and Δt is the time until the option hedge is closed out.⁴ Equation (21) shows that the investor has, in effect, paid $\alpha(w/w_1)$ plus continuous rebalancing

costs to save the transaction costs on the amount by which x^* might exceed c at expiration (i.e., w^*). Under reasonable assumptions this will involve a net outflow, but the costs will be small relative to the size of the hedge. Moreover, these costs relate to more than one option position. When the hedge begins it consists of $1/w_1$ options and if there are transaction cost savings at the dissolution of the hedge it is because the option is in the money at expiration and therefore has a hedge ratio of one (i.e., equation (21) then relates to only one option).

Similarly, a borrowing hedge can be formed without the marginal cost of stock sales and purchases. In this case an investor who wishes to purchase a stock does not purchase it immediately, instead he buys the usual hedged position of w/w_1 options and continuously rebalances as if he actually held the stock. Eventually one of two things happens:

- 1) The option price drops below $1/16$ point at which time trading is halted. A hedge can no longer be formed and the stock is finally, actually, purchased.
- 2) The option is in the money at expiration at which time the option is exercised and the stock is (finally) acquired.

Since transaction costs are the same for buying and selling, equation (21) will also describe the investor's costs for a borrowing hedge. The investor has, in effect, paid $\alpha(w/w_1)$ per share plus continuous rebalancing costs to postpone the cost of acquiring the stock and save $\alpha_x w^*$ (per share) when the stock is finally acquired.

Since the continuously rebalanced option hedge position mimics every price movement of the underlying stock, the investor has, in effect bought the stock immediately without paying for the stock until the option expires or becomes worthless. This procedure is therefore

a substitute for margin borrowing but without a margin requirement (or collateral of any kind).⁵

IV. The Combined Effects of Transaction Costs and Different Borrowing and Lending Rates

The effects of transaction costs and different borrowing and lending rates are presented in Table 1. Column (1) is the Black-Scholes option price calculated under the assumptions specified in the table. Column (9) is the value the Black-Scholes model gives if the specified borrowing rate is substituted for the risk free rate.

Columns (2) and (8) ("Rebal Adj") are the difference between the traditional B&S option price and the continuous rebalancing transaction cost price derived in Section II. In addition to the assumptions specified in the table, the continuous transaction cost price assumes that one way transaction costs for options are 3% and the underlying stock's expected return is 17% per year.

INSERT TABLES 1-3 ABOUT HERE

Columns (3) and (7) ("Init-End Adj") are estimates of the adjustment to the option price required to cover the cost of acquiring the hedge and finally liquidating it. These costs are based on the trading techniques presented in the previous section and embodied in equation (21). Common stock transaction costs (α_x) are assumed to be 1.5% and option transaction costs (α) are assumed to be 3%.⁶ The expected value of the future value of the option is calculated from Sprenkle's equation (see Smith, 1976, page 17) and this value is discounted back to the present using a discount rate, K , derived from the CAPM under the assumptions that the market return is 17% per year, the risk free rate is 12%, the

The Effects of Transaction Costs and Different Borrowing and Lending Rates on Option Prices

ANNUAL STD. DEV. OF C.S. RETURNS = .3

		Investing at 12%				Borrowing at 15%				
Stock Price	Time To Exp (Months)	B&S Price (1)	Rebal Adj (2)	Init -End Adj (3)	Net Adj Price (4)	Net Price Spread (5)	Net Adj Price (6)	Init -End Adj (7)	Rebal Adj (8)	B&S Price (9)
28	1	.000	.000	.000	.000	-.000	.000	-.000	-.000	.000
	5	.152	.005	.008	.166	-.005	.161	-.009	-.005	.176
	9	.655	.017	.030	.702	.010	.712	-.034	-.016	.763
34	1	.050	.002	.003	.054	-.006	.049	-.003	-.002	.054
	5	1.208	.023	.047	1.278	-.019	1.259	-.051	-.022	1.331
	9	2.508	.040	.082	2.629	.044	2.673	-.090	-.036	2.799
40	1	1.583	.017	.044	1.643	-.069	1.574	-.045	-.016	1.636
	5	4.085	.038	.102	4.225	-.009	4.215	-.108	-.034	4.358
	9	5.911	.049	.139	6.100	.109	6.208	-.150	-.043	6.401
46	1	6.461	.006	.103	6.570	-.126	6.444	-.105	-.006	6.555
	5	8.650	.033	.159	8.842	-.001	8.841	-.168	-.028	9.038
	9	10.538	.044	.199	10.781	.148	10.929	-.213	-.037	11.179
52	1	12.399	.000	.186	12.585	-.277	12.308	-.189	-.000	12.498
	5	14.144	.017	.227	14.388	-.052	14.336	-.238	-.014	14.588
	9	15.897	.031	.265	16.193	.128	16.320	-.282	-.025	16.627

TABLE 2

The Effects of Transaction Costs and Different Borrowing and Lending Rates on Option Prices

ANNUAL STD. DEV. OF C.S. RETURNS = .3

		Lending at 12%				Borrowing at 15%				
Stock Price	Time To Exp (Months)	B&S Price (1)	Rebal Adj (2)	Init -End Adj (3)	Net Adj Price (4)	Net Price Spread (5)	Net Adj Price (6)	Init -End Adj (7)	Rebal Adj (8)	B&S Price (9)
28	1	.049	.002	.003	.053	-.007	.046	-.003	-.002	.050
	5	1.525	.028	.062	1.615	-.104	1.511	-.065	-.028	1.603
	9	3.166	.047	.109	3.322	-.130	3.191	-.114	-.044	3.350
34	1	.660	.014	.029	.702	-.069	.633	-.030	-.014	.676
	5	3.736	.048	.118	3.902	-.184	3.718	-.122	-.047	3.887
	9	6.022	.066	.170	6.258	-.187	6.070	-.177	-.062	6.309
40	1	2.950	.030	.082	3.063	-.177	2.886	-.083	-.030	2.999
	5	7.017	.060	.178	7.255	-.252	7.003	-.183	-.058	7.244
	9	9.683	.074	.232	9.989	-.232	9.758	-.241	-.070	10.068
46	1	7.188	.028	.138	7.354	-.256	7.098	-.140	-.027	7.265
	5	11.175	.063	.239	11.477	-.312	11.165	-.245	-.059	11.470
	9	13.974	.078	.296	14.349	-.280	14.069	-.307	-.071	14.447
52	1	12.592	.015	.203	12.810	-.346	12.465	-.205	-.014	12.684
	5	15.975	.060	.302	16.337	-.378	15.959	-.310	-.055	16.323
	9	18.739	.078	.362	19.179	-.341	18.838	-.374	-.072	19.284

TABLE 3

The Effects of Transaction Costs and Different Borrowing and Lending Rates on Option Prices

ANNUAL STD. DEV. OF C.S. RETURNS = .9

		Investing at 12%				Borrowing at 15%			
Stock Price	Time To Exp (Months)	B&S Price (1)	Rebal Adj (2)	Init		Net Price Spread (5)	Net		B&S Price (9)
				-End Adj (3)	Net Adj Price (4)		Adj Price (6)	-End Adj (7)	
28	1	.364	.009	.018	.391	-.048	.343	-.019	.371
	5	3.513	.049	.119	3.682	-.240	3.442	-.122	3.612
	9	6.045	.064	.177	6.286	-.291	5.995	-.182	6.238
34	1	1.635	.027	.062	1.723	-.154	1.569	-.063	1.658
	5	6.358	.065	.181	6.604	-.345	6.260	-.185	6.508
	9	9.464	.079	.243	9.786	-.388	9.398	-.249	9.724
40	1	4.316	.041	.118	4.475	-.274	4.201	-.120	4.362
	5	9.926	.075	.245	10.246	-.442	9.805	-.249	10.126
	9	13.403	.089	.310	13.802	-.476	13.326	-.317	13.726
46	1	8.332	.046	.176	8.554	-.378	8.176	-.178	8.399
	5	14.072	.085	.309	14.467	-.546	13.921	-.315	14.317
	9	17.750	.094	.379	18.222	-.568	17.654	-.387	18.130
52	1	13.288	.037	.237	13.563	-.469	13.094	-.239	13.370
	5	18.664	.082	.375	19.121	-.633	18.488	-.382	18.948
	9	22.415	.101	.448	22.964	-.669	22.295	-.458	22.846

beta of the underlying stock is one and the beta of the option (as pointed out by Black and Scholes, 1972) is:

$$\beta_w = xw_1\beta_x/w$$

Columns (4) and (6) ("Net Adj Price") are Black & Scholes option prices (columns (1) and (7)) with both types of transaction costs added (for the investment hedge, columns (2) and (3)) or subtracted (for the borrowing hedge, columns (7) and (8)). Columns (4) and (6) therefore show the prices the options must sell for to net the specified borrowing and lending rates after transaction costs.⁷ Needless to say, these transaction costs would be different for different sets of assumptions. The costs presented are illustrative and can be helpful in understanding the nature of the phenomena. The reader is encouraged to analyse the effects of his own assumptions.⁸

Column (5) ("Net Price Spread") is the result of subtracting column (4) from column (6). It should be interpreted as follows:

- 1) Negative values indicate that the option price is the bounded range of prices between the prices specified in columns (4) and (6). For option prices within this range neither investment hedges nor borrowing hedges are particularly attractive. The reader will note that the highest option price which produces an attractive borrowing hedge (Column (6), Net Adj Price - Borrowing) is frequently higher than the traditional Black-Scholes price (Column (1)). When this occurs, the Black-Scholes price isn't even a legitimate equilibrium value. Under these conditions if an option actually sells for the Black-Scholes price excess profits can be made by forming a borrowing hedge and borrowing at below the market rate.
- 2) A zero value indicates a unique equilibrium option price (i.e., the value shown in both column (6) and column (4)). This occurs when transaction cost effects and borrowing and lending effects precisely cancel. This unique option price is never the Black-Scholes price except in the trivial case where all prices are equal to zero.

- 3) A positive value indicates that the option hedge can be viewed as a financial intermediary with a lower spread than traditional intermediaries. If the option sells at a price between the prices specified in columns (4) and (6) the option hedge is simultaneously a higher return risk free investment than treasury bills and a lower cost source of funds than traditional borrowing. When this amazing situation occurs there is no price to which the option can adjust which will eliminate excess profits. For example, if the option price were to drop low enough for the investment hedge to no longer be attractive, this would only make a borrowing hedge even better. It may be that the only thing that prevents all short term borrowing and lending from being sucked into these financial "black holes" is the limited number of investors in the special transaction cost situations described in the previous section.

This permanent disequilibrium is bounded between the column (6) and column (4) prices. If the option price is above the column (6) price the option is not attractive as a part of a borrowing hedge but a short position in the option will be very desirable as part of an investment hedge. This unbalanced selling pressure should drive the option price below the column (6) price at which time the option is desirable as both an investment hedge and a borrowing hedge. This presumably results in a better balance between supply and demand for the option.

Similarly, if the option were to sell below the column (4) price it would be very attractive as a part of a borrowing hedge but there would be no interest in forming investment hedges. The resulting net buying pressure should push the option price back above the column (4) value.

Clearly, when the column (5) value is positive, it indicates a bizarre form of bounded disequilibrium.

V. Conclusion

This paper analyzes the effects of transaction costs and different borrowing and lending rates on option pricing. A continuous time option pricing model which includes transaction costs resulting from continuous hedge rebalancing is derived.

The paper also suggests ways in which some classes of investors can minimize the cost of initially acquiring and ultimately disposing of the hedge position.

When transaction costs and different borrowing and lending rates are taken into account three situations seem to arise (depending on market conditions and option characteristics):

- 1) Occasionally the effect of different borrowing and lending rates precisely offsets the effects of transaction costs to yield a unique option price. This price is never the Black-Scholes price.
- 2) Usually transaction cost effects and borrowing and lending cost effects partially offset each other yielding a bounded range of option prices. The center of the range is never the Black-Scholes price and the bounded range of prices frequently fails to include the B-S price as a legitimate equilibrium value. When this occurs, excess profits can be achieved if options actually trade at the B-S price.
- 3) On some occasions the borrowing and lending rate effect seems to be larger than the transaction cost effect. When this occurs, the option hedge becomes society's lowest cost financial intermediary. Risk free investments can be made at more than the risk free rate and borrowing can be conducted at less than the borrowing rate.

Appendix A

The Implications of Hedge Rebalancing Using Adjustments to the Option Position

Throughout this paper the authors assume that rebalancing is done by adjusting the option portion of the hedge. This is generally the cheapest way to rebalance because option rebalancing involves smaller dollar amounts than rebalancing with common stock. This cost advantage is partially offset by the fact that the average bid-ask spread in the options market is greater than in the stock market (see Phillips and Smith (1980)).

The hedge acquisition and dissolution techniques described in the text assume that the hedge contains the same number of shares of stock at the beginning and end of the life of the hedge. Therefore, equation (24) will only (usually) be an accurate description of costs if all rebalancing is done with options (thus leaving the number of shares in the hedge unchanged throughout the life of the hedge).

Needless to say, an investor should not rebalance by buying an overpriced option or selling an underpriced option. Therefore, the assumption that all rebalancing is done with options is unrealistic.

However, one suspects that the advantage of being able to rebalance with options when they are favorably priced and avoid them (with stock rebalancing) when they are unfavorably priced probably more than offsets the additional rebalancing cost of common stock rebalancing and the additional hedge dissolution cost which may result from acquiring an unwanted common stock position to liquidate at the end of the life of the hedge. Moreover, common stock rebalancing can also result in

the acquisition of part of the desired stock position prior to dissolution thus reducing costs below those assumed in equation (24).

Moreover, footnote 6 shows how some investors can reduce transaction costs to a level generally below those presented in this paper. Finally, the reader may feel that the option rebalancing assumption is unrealistic because it is not possible to trade options in odd lots. The authors suggest that this is not a real problem because, if the investor's hedge is so small that rebalancing involves trades of less than several thousand dollars each, transaction costs will destroy the investor no matter how he rebalances.

Appendix B

A Demonstration of the Validity of the Proposed Solution to Equation (9).

Even the most sophisticated reader will find it virtually impossible to verify the accuracy of the computer program presented in Appendix ^C~~B~~. Although the program is in Fortran IV, the operating system (Control Data Cyber 175 as modified locally) and the math utilities packages will be unfamiliar.

The authors therefore present this example in hopes of convincing the reader that the proposed solution to equation (6) is correct. In order to provide a simple example, the authors have chosen parameters which are realistic but relatively easy to calculate:

$$x = \$50$$

$$c = \$50$$

$$v^2 = .25$$

$$(t^* - t) = .5 \text{ (years)}$$

$$r = .10$$

$$\delta = .15$$

$$\alpha = .02$$

The g function (equation (11)) on the right side of equation (6) is the instantaneous dollar amount of rebalancing required. It can be calculated from Black & Sholes pricing theory based on the parameters listed above. This yields:

$$w = 8.1316$$

$$d_1 = \left(\ln \frac{x}{c} + (r + .5v^2)(t^* - t) \right) / (v^2(t^* - t))^{1/2} = .31820$$

$$w_1 = N(d_1) = .62483$$

For simplicity define

$$m = d_1 / (2)^{1/2} = .225$$

w_{11} can then be expressed as:

$$\begin{aligned} w_{11} &= e^{-m^2} / (xv(2\pi(t^* - t))^{1/2}) \\ &= .02145 \end{aligned}$$

Also:

$$\begin{aligned} w_{12} &= ((\ln \frac{x}{c}) / (t^* - t) - r - \frac{v^2}{2})(e^{-m^2}) / (2v(2\pi(t^* - t))^{1/2}) \\ &= -.12068 \\ w_{111} &= -e^{-m^2} (v / (2(t^* - t))^{1/2} + m / (t^* - t)) / (v^2 x^2 \pi^{1/2}) \\ &= -.00081523 \end{aligned}$$

Substituting these values into equation (11) of the paper yields:

$$g = 4.625$$

The right side of equation (6) is therefore:

$$\alpha g = .0925$$

The left side of equation (6) includes partial derivatives of the continuous transaction cost option price derived in this paper. These derivatives must be approximated by taking small interval values about the \$50 stock price and the .5 year time to expiration:

Parameter	Estimation Interval	Estimated Value
w	----	8.1803
w ₁	\$50 $\pm$ \$.30	.6270
w ₁₁	\$50 $\pm$ \$2.00	.02119
w ₂	.5 year $\pm$.005 year	-9.0358

The width of the interval (column 2 above) used to approximate each partial derivative is a function of the 5 to 6 significant digit accuracy of the option price, w, as calculated from the computer program listed in Appendix C.

When the estimated values from column 3 above are substituted into the left side of equation (6) they yield .0953. Considering the inherent inaccuracy of small interval approximations, this seems to be a good approximation of the value previously calculated for the right side of equation (6) (i.e., $\alpha g = .0925$).

Footnotes

¹Thorpe (1973) demonstrated that option hedges can be sources of funds despite restrictions on short selling.

²These results are derived under the assumption that rebalancing is done by buying or selling options (rather than stock). See Appendix A for a discussion of the implications of this assumption.

³When an option is exercised the commission is based on the exercise price not the stock price. Therefore the investor pays $\alpha_x \text{Min}(x^y; c)$ when the hedge is terminated.

⁴(21) can usually be usefully approximated by

$$\alpha w/w_1 - \alpha_x w$$

$$(i.e., e^{-K\Delta t} E(w^*) \approx w).$$

Smith (1976) and others have pointed out that the Black-Scholes value of w is the present value at the risk free rate of the expected value of the option at maturity if the growth rate of the stock is assumed equal to the risk free rate. w will understate the expected value of the option at expiration whenever the expected return of the stock exceed the risk free rate but w also understates the appropriate discount rate by which the present value of $E(w^*)$ should be calculated if the beta of the option is greater than zero.

The approximation $e^{-K\Delta t} E(w^*) \approx w$ is usually satisfactory because the two effects partially offset each other and neither effect is (usually) overwhelmingly important.

⁵One problem with the Black Scholes model is its dependence on the assumption that short positions in common stock are an immediate source of funds. Thorpe (1973) has argued that if an investor currently owns the stock for which a hedge is to be created, selling the stock is equivalent to short selling and is a source of funds. The procedure for forming a minimum transaction cost borrowing hedge described in this paper is another way in which a borrowing hedge can be a source of funds.

⁶Phillips and Smith (1980) point out that transaction costs for both stocks and options are about 1%. P&S also point out that the bid-ask spread is about 1% for stock but about 4% for options. This paper assumes one way transaction costs of 1/2 the bid-ask spread plus the normal transaction costs. This, in effect, assumes there is a single "true" pre-transaction cost stock or option price half way between the bid and ask. The bid-ask spread is therefore considered to be just another part of transaction costs.

⁷In effect Tables 1, 2 and 3 assume that the costs of initiating and terminating the option hedge are prepaid by adding or subtracting them from the initial option price. This change in initial option price will slightly alter rebalancing transaction costs and is not taken into consideration in the derivation of the continuous rebalancing model (equations (1) through (19)).

Given the generally small size of initiation and termination costs relative to the option price, and the approximate nature of other aspects of the option pricing model, this should be acceptable for most purposes.

⁸Some investors will be able to reduce transaction costs to levels below those assumed in this paper by rebalancing less frequently. For a borrowing hedge using the acquisition and dissolution technique suggested in the paper, a failure to rebalance continuously means that the equivalent number of shares being mimicked by the option position changes slightly as the hedge ratio changes but is not immediately rebalanced (i.e., the investor may have the option equivalent of 96 shares rather than the 100 shares he originally intended, etc.). Some investors may not care about the exact number of common stock equivalents his option position equals. If not, he can save money by rebalancing only after the hedge ratio has changed by a predetermined amount. He can then avoid the transaction costs resulting from all of the reversals of the hedge ratio occurring between his finite hedge ratio limits. Moreover, this strategy allows the investor to save money by rebalancing in larger amounts.

The same periodic rebalancing strategy can be applied to the investment hedge strategy described in the paper except that in this case a failure to rebalance means that the investor will have a small long or short position in the stock he originally wished to sell (i.e., instead of having the equivalent of no stock, his equivalent position might fluctuate between long and short positions amounting to a few percent of his original holding.)

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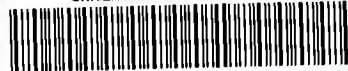
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